Ellipsis of tolerance

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1 probabilistic modelling of the problem

One puts[1] X a random gaussian bivariate vector

- of hope $\mu=(\mu_1, \mu_2) = E(X)$
- of variance/covariance matrix : $\Sigma = E(X_i - \mu)(X_i - \mu)^T$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

One has therefore $X =\sim N_2(\mu,\Sigma)$ The μ and Σ parameters are supposed unknown. One estimates μ by the estimator of the verisimilitude maximum (MLE) :

$$G = \frac{\Sigma_{1<i<n} X_i}{n}$$

where the $X_i \sim \mathcal{N}_2(\mu, \Sigma)$ are independent gaussian vectors identically distributed.

In the same way, one estimates the matrix Σ by the estimator of the verisimilitude maximum (skewed estimator) :

$$V = \frac{1}{n} \sum_{1<i<n} (X_i - G)(X_i - G)^T$$

2 Some Properties

One puts $X = (x_1, x_2)$, one has the following results then:

- the density function of the law $\mathcal{N}_2(\mu, \Sigma)$:

[1] In this document, the random variables are bold written

$$f(X) = \frac{1}{(2\pi)\sqrt{\det\Sigma}} \exp\left(\left(-\frac{1}{2}(X-\mu)\right)^T \Sigma^{-1}(X-\mu)\right)$$

$$f(X) = \frac{1}{(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2p\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}]\}$$

the square of the Mahalanobis distance follows the law of Fisher[2] to 2 and n—2 degrees of freedom:

$$\frac{n(n-2)}{2(n-1)(n+1)}(X-G)^T V^{-1}(X-G) \sim F(2,n-2)$$

3 90% ellipsis of tolerance

While putting

$$r^2 = \frac{2(n-1)(n+1)}{n(n-2)} Q_{f(2,n-2)}(0.9)$$

where $Q_{\mathcal{F}}(2, n—2)$ is the quantile function of the Fisher law, one gets the equation of the 90% tolerance ellipsis:

$$(X-G)^T V^{-1}(X-G) = r^2$$

One does the diagonalisation of the V matrix. One has $V = PDP^{-1}$. D is the diagonal matrix containing the eigen values λ_I and λ_2. λ_I is the inertia explained by the main axis (variance of the main axis), and λ_2 is the inertia explained by the small axis (variance of the small axis). And P is the matrix of basis change. The P matrix is orthogonal ($P^{-1} = p^T$). One has then

$V^{-1}=PD^{-1}P^{-1}=PD^{-1}P^T$ From where

$$(X\text{-}G)^T V^{-1}(X\text{-}G)=(X\text{-}G)^T PD^{-1} P^T(X\text{-}G)=(P^T(X\text{-}G))^T D^{-1}(P^T(X\text{-}G))$$

One puts then $Y = P^T (X—G)$ and one gets:

$$Y^T D^{-1} Y=r^2$$

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2 See the book of Gilbert SAPORTA " Probabilités, analyse de données et statistique", second edition, page 316, section 13.6.2

ie for $Y = (y_1, y_2)$

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = r^2$$

If $y_2 = 0$ then

$$y_1 = \pm\sqrt{r^2\lambda_1}$$

And in the same way,

$$y_2 = \pm\sqrt{r^2\lambda_2}$$

4 application

For n = 40*60 = 2400, one gets:

$$y_1 = \pm\sqrt{4.613 * \lambda_1}$$

For $y_1 = 0$, one has

$$y_2 = \pm\sqrt{4.613 * \lambda_2}$$

The radius of each axis is given therefore by 2.14 times the standard-deviation

$$\boxed{(\sqrt{\lambda_1}) ou (\sqrt{\lambda_2})}$$

of the projection on the axis.